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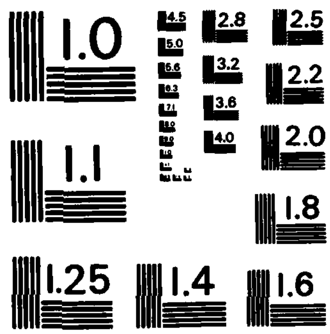
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NONLINEAR CAPILLARY WAVES UNDER GRAVITY WITH EDGE CONSTRAINTS IN A CHANNEL

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MATHEMATICS RESEARCH CENTER

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ABSTRACT

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AMS (MOS) Subject Classification: 76B15

Key Words: equation of the K-dV type, capillary waves, edge constraints, channel of variable cross section, ray method

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SIGNIFICANCE AND EXPLANATION

An equation of the K-dV type is derived for the study of nonlinear surface waves under gravity with surface tension and edge constraints in a channel of variable cross section. The approach used here is based upon a nonlinear ray method. An example of a symmetric rectangular channel with variable width is given.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

NONLINEAR CAPILLARY WAVES UNDER GRAVITY
WITH EDGE CONSTRAINTS IN A CHANNEL

M. C. Shen*

§1. INTRODUCTION

The dynamical problem of capillary waves on a liquid in a channel has not been much studied in the past. The difficulty may lie in the fact that the edge condition at the line of contact of the liquid free surface with the channel wall so far remains an open problem. It is well known that for water at rest the contact angle at the edge of the free surface is constant. However, when the surface is in motion, the edge condition depends upon whether the channel wall is wetted or not and has to be determined as a part of the solution to the free surface problem. In our study of capillary waves under gravity on a viscous fluid in an inclined channel (Shih, 1973; Shen and Shih, 1974), we made the assumption that the fluid velocity should be continuous at the line of contact. It follows that the free surface elevation will remain zero if initially it is zero there. This argument would seem to be plausible only for small amplitude waves. By a different consideration the same edge condition was also used by Kopachevskii (1967).

In two recent papers by Benjamin and Scott (1979) and Benjamin (1981), the problem of gravity-capillary waves with edge constraints on an inviscid fluid has been investigated both experimentally and theoretically. They considered a straight horizontal channel of uniform cross section, and filled the channel with water up to the brim so that the edge condition of zero surface elevation can be experimentally realized. In the former paper, the linear problem was studied. In the latter, variational principles were formulated for the full nonlinear equations as a Hamiltonian system and for waves of permanent type. However, as pointed out by Benjamin (1981), a rigorous treatment of the nonlinear problem is beyond reach at present. In this paper, we shall use a different approach to study a

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more general problem, that is, the channel is allowed to have a slowly varying cross section. The main purpose of our work is to derive an equation of the K-dV type, which may be used as an approximate equation for the investigation of long capillary waves with edge constraints in a channel.

Our approach is based upon the nonlinear ray method developed in Shen and Keller (1973). The basic ideas may be explained as follows. The motion of a wave front is assumed to be determined by a phase function. It is found that the phase function satisfies the Hamilton-Jacobi equation in geometrical optics, the solutions of which determine a family of bicharacteristics called rays. An equation of the K-dV type is then derived along the rays for the wave amplitude. In a report by Zhong and Shen (1982), we specialized the procedure in Shen and Keller (1973) to the case of an incompressible inviscid fluid without surface tension, and an equation of the K-dV equation for a channel of variable cross section was derived. The present work is an extension of the previous result and a host of problems, such as fission of solitons (Johnson, 1973; Zhong and Shen, 1982), shelf generation behind a solitary wave (Knickbocker and Newell, 1980) and others, may be studied similarly by means of the equations derived here. However, this report is self-contained, there is minimum reference to the previous report. We also note that our method is related to those developed in Kuzmak (1959) and Choquet-Bruhat (1969).

We formulate the problem in §2. In §3 we derive an equation of the K-dV type for a general channel. In §4 the example of a symmetric rectangular channel with variable width is considered to illustrate the method. Finally some discussions are given in §5.

§2. FORMULATION OF THE PROBLEM

We consider the motion of an inviscid, incompressible fluid of constant density with surface tension in a channel with a smooth convex boundary defined by $h^*(x^*,y^*,z^*) = 0$, where z^* is positive upward (Figure 1). The lines of contact are two curves of intersection of $h^* = 0$ with the plane $z^* = d^*$. The governing equations are

$$u^*_x + v^*_{y^*} + w^*_{y^*} = 0 , \tag{1}$$

$$u^*_{t^*} + u^*u^*_{x^*} + v^*u^*_{y^*} + w^*u^*_{z^*} = -p^*_{x^*}/\rho^* , \tag{2}$$

$$v^*_{t^*} + U^*_{x^*} + v^*v^*_{y^*} + w^*v^*_{z^*} = -p^*_{y^*}/\rho^* , \tag{3}$$

$$w^*_{t^*} + u^*w^*_{x^*} + v^*w^*_{y^*} + w^*w^*_{z^*} = -p^*_{z^*}/\rho^* - g^* , \tag{4}$$

subject to the boundary conditions: At

$$\begin{gathered} z^* = \eta^*(x^*,y^*,t^*) , \\ \eta^*_{t^*} + u^*\eta^*_{x^*} + \eta^*_{y^*} - w^* = 0 , \end{gathered} \tag{5}$$

$$\begin{gathered} p^* = -T^*[\eta^*_{x^*x^*}(1 + \eta^{*2}_{y^*}) + \eta^*_{y^*y^*}(1 + \eta^{*2}_{x^*}) - 2\eta^*_{x^*y^*}\eta^*_{x^*}\eta^*_{y^*}] \times \\ (1 + \eta^{*2}_{x^*} + \eta^{*2}_{y^*})^{-3/2} ; \end{gathered} \tag{6}$$

at $h^*(x^*,y^*,z^* = d^*) = 0$,

$$\eta^* = d^* ; \tag{7}$$

at $h^*(x^*,y^*,z^*) = 0$,

$$u^*h^*_{x^*} + v^*h^*_{y^*} + w^*h^*_{z^*} = 0 . \tag{8}$$

Here (u^*,v^*,w^*) is the velocity, t^* is the time, g^* is the constant gravitational acceleration, ρ^* is the constant density, p^* is the pressure, $z^* = \eta^*$ is the equation of the free surface and T^* is the constant surfact tension coefficient. Within the framework of long wave approximation, we assume that the channel bottom varies slowly in the longitudinal direction and the magnitude of the transverse velocities is much smaller than that of the longitudinal velocity. Under these assumptions, we introduce the nondimensional variables

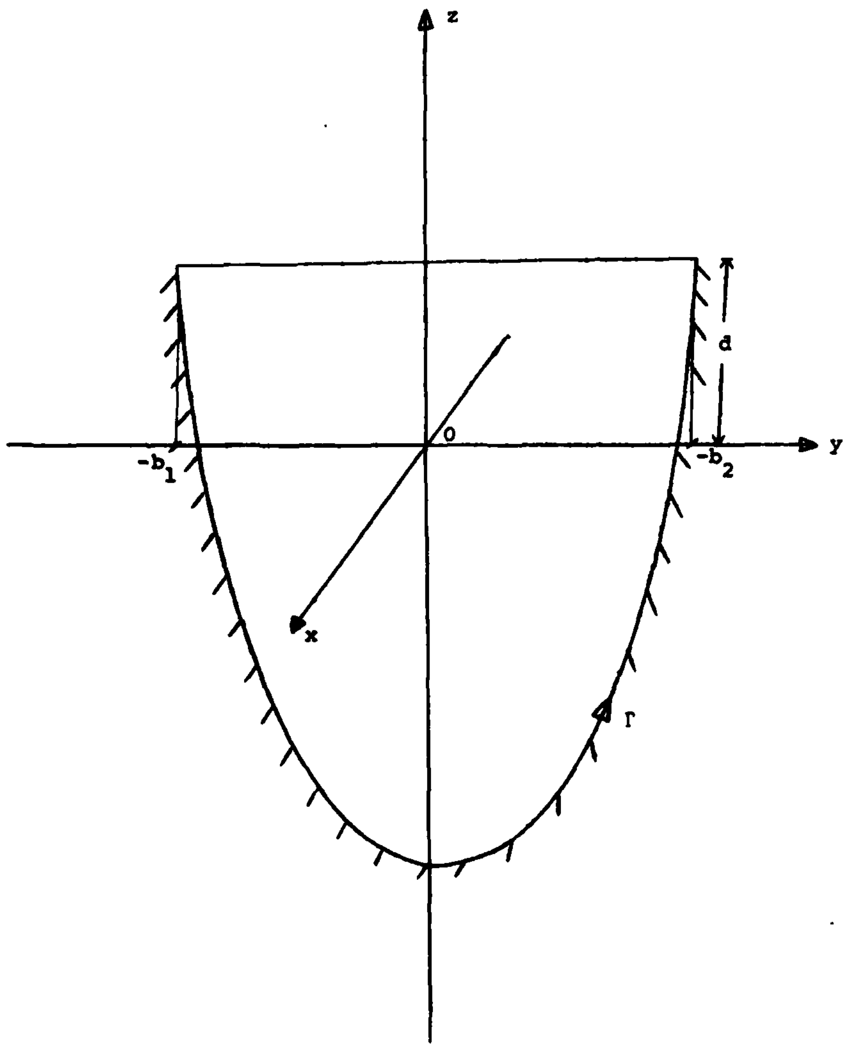

Figure 1. A cross section of the channel.

$$t = \beta^{-3/2} t^*/(H/g)^{1/2}, \quad (x,y,z) = (\beta^{-3/2} x^*/H, y^*/H, z^*/H) .$$

$$\eta = \eta^*/H, \quad h = h^*/H, \quad p = p^*/(\rho g H) .$$

$$(u,v,w) = (u^*/(gH)^{1/2}, \beta^{1/2} v^*/(gH)^{1/2}, \beta^{1/2} w^*/(gH)^{1/2}) ,$$

$$T = T^*/(\rho g H^2), \quad \beta^{3/2} = L/H >> 1, \quad d = d^*/H ,$$

where L,H are respectively the longitudinal and transverse length scales. In terms of the unstarred variables, (1) to (8) become

$$u_x + \beta(v_y + w_z) = 0 , \tag{9}$$

$$u_t + uu_x + \beta(vu_y + wu_z) = -p_x , \tag{10}$$

$$v_t + uV_x + \beta(vu_y + wu_z) = -\beta^2 p_y , \tag{11}$$

$$w_t + uw_x + \beta(vw_y + Ww_z) = -\beta^2(p_z + 1) , \tag{12}$$

subject to the boundary conditions:

$$\begin{aligned} &\text{At} \quad z = \eta(x,y,t) , \\ &\qquad \eta_t + u\eta_x + \beta(v\eta_y - w) = 0 , \end{aligned} \tag{13}$$

$$\begin{aligned} p = &-T[\beta^{-3}\eta_{xx}(1 + \eta_y^2) + \eta_{yy}(1 + \beta^{-3}\eta_x^2) \\ &-2\beta^{-3}\eta_{xy}\eta_x\eta_y](1 + \beta^{-3}\eta_x^2 + \eta_y^2)^{-3/2} ; \end{aligned} \tag{14}$$

$$\begin{aligned} &\text{at} \quad h(x,y,z,z = d) = 0 , \\ &\qquad \eta = d ; \end{aligned} \tag{15}$$

$$\begin{aligned} &\text{at} \quad h(x,y,z) = 0 , \\ &\qquad uh_x + \beta(vh_y + wh_z) = 0 . \end{aligned} \tag{16}$$

§3. DERIVATION OF THE EQUATION OF THE K-dV TYPE

We introduce a phase function $S = S(t,x)$ and let $\xi = \beta S$. Assume that u,v,w,p and η also depend upon ξ and possess an asymptotic expansion of the form

$$\phi(\xi,t,x,y,z,\beta) \sim \phi_0 + \beta^{-1}\phi_1 + \beta^{-2}\phi_x + \dots , \tag{17}$$

where ϕ_0 is given by

$$(u_0,v_0,w_0) = 0, \quad p = -z + d, \quad \eta_0 = d . \tag{18}$$

Substitution of (17) in (9) to (16) will yield a sequence of equations and boundary conditions. The equations for the first approximation are

$$ku_{1\xi} + v_{1y} + w_{1z} = 0 , \tag{19}$$

$$-\omega u_{1\xi} + kp_{1\xi} = 0 , \tag{20}$$

$$p_{1y} = p_{1z} = 0 , \tag{21}$$

subject to the boundary conditions

$$\omega\eta_{1\xi} + w_1 = 0 \tag{22}$$

$$p_1 = \eta_1 - T\eta_{1yy} \quad \text{at} \quad z = d, \quad -b_1(x) < y < b_2(x) , \tag{23}$$

$$\eta_1 = 0 \quad \text{at} \quad z = d, \quad y = -b_1(x), b_2(x) , \tag{24}$$

$$v_1 h_y + w_1 h_z = 0 \quad \text{at} \quad h = 0 , \tag{25}$$

where $\omega = -S_t$, $k = S_x$, and $y = -b_1(x)$, $y = b_2(x)$ are lines of contact. From (21), we see that $p_1 = p_1(t,x,\xi)$. To solve (23), (24) for η_1, let

$$\eta_1 = \nu_1(x,y)p_1 , \tag{26}$$

and $\nu_1(x,y)$ satisfies

$$\nu_{1yy} - \frac{1}{T}\nu_1 = -1/T, \quad -b_1 < y < b_2 ,$$

$$\nu_1 = 0, \qquad y = -b_1, b_2 .$$

It is easily found that the solution for ν_1 is

$$\nu_1 = 1 - [\cosh\mu(y - (b_2 - b_1)/2)]/\cosh(\mu b/2) \tag{27}$$

where $\mu = \left(\frac{1}{T}\right)^{1/2}$, and $b = b_1 + b_2$. We integrate (19) over a cross section D of the channel and make use of (20), (22) and (25) to obtain

$$p_{1\xi}[a(x)k^2 - \omega^2 \int_{-b_1}^{b_2} \nu_1(x,y)dy] = 0 ,$$

where $a(x)$ is the area of the cross section. Suppose $p_{1\xi} \not\equiv 0$. Then we have

$$\omega/k = \pm[a/b(1 - (2/\mu b)\tanh\mu b/2)]^{1/2} = G(x) , \tag{28}$$

which is in the form of Hamilton-Jacobi equatin in geometrical optics, and can be solved by the method of characteristics. The characteristic equations are

$$\begin{aligned} dt/d\sigma = \mu,\quad dx/d\sigma = \mu G,\quad dk/d\sigma = -k\mu G' , \\ d\omega/d\sigma = dS/d\sigma = 0 , \end{aligned} \tag{29}$$

where μ is a proportionality factor. The solutions of (29) determine a one-parameter family of bicharacteristics, called rays. As seen from (29), both ω and S are constant along a ray. Let $\sigma = t$ in (31) and from $dx/dt = G(x)$ we obtain by integration the equation of a ray

$$\int_{x_0}^{x} [G(\zeta)]^{-1} d\zeta = t - t_0 .$$

where (t_0,x_0) is the initial point of a ray. We may choose $x_0 = 0$ and prescribe $S = -t$ on $x = 0$. Then

$$S = -t_0 = -t + \int_0^x [G(\zeta)]^{-1} d\zeta . \tag{30}$$

It follows that

$$\omega = 1,\quad k = [G]^{-1} . \tag{31}$$

Needless to say, other choices of S are also possible.

The equations for the second approximation are

$$kU_{2\xi} + V_{2y} + W_{2z} + U_{1x} = 0 , \tag{32}$$

$$-\omega u_{2\xi} + ku_,u_{1\xi} + kp_{2\xi} + u_{1t} + p_{1x} = 0 , \tag{33}$$

$$p_{2y} = \omega v_{1\xi} , \tag{34}$$

$$p_{2z} = \omega w_{1\xi} , \tag{35}$$

subject to the boundary conditions

$$-\omega\eta_{2\xi} + ku_1\eta_{1\xi} - w_2 - w_{1z}\eta_1 + \eta_{1t} = 0 \ , \tag{36}$$

$$p_2 - \eta_2 = -T(\eta_{1\xi\xi} + \eta_{2yy}) \quad \text{at} \quad z = d, \quad -b < y < b_2 \tag{37}$$

$$\eta_2 = 0 \quad \text{at} \quad z = d, \quad y = -b_1, b_2 \tag{38}$$

$$v_2 h_y + w_2 h_z = -u_1 h_x \quad \text{at} \quad h = 0 \ . \tag{39}$$

By differentiating (34), (35) with respect to y and z respectively, adding, and making use of (19) and (20), we obtain

$$\nabla^2 p_2 = -k^2 p_{1\xi\xi} \ . \tag{40}$$

Then we differentiate (22) and (25) with respect to ξ and make use of (34) and (35) to obtain

$$p_{2z} = -\omega^2 \nu_1 p_{1\xi\xi} \quad \text{at} \quad z = d, \quad -b_1 < y < b_2 \ , \tag{41}$$

$$p_{2y} h_y + h_{2z} h_z = 0 \quad \text{at} \quad h = 0 \ . \tag{42}$$

Let

$$p_2 = -\ \phi(t,x,y,z) p_{1\xi\xi} + A_2(t,x,\xi) \ . \tag{43}$$

(40) to (42) imply

$$\nabla^2 \phi = k^2 \quad \text{in} \quad D \ , \tag{44}$$

$$\phi_z = \omega^2 \nu_1 \quad \text{at} \quad z = d, \quad -b_1 < y < b_2 \ , \tag{45}$$

$$\phi_y h_y + \phi_z h_z = 0 \quad \text{at} \quad h = 0 \ . \tag{46}$$

The Neumann problem posed by (44) to (46) is solvable as a consequence of (28). If we assume $u_1, v_1, w_1, p_1, p_{1\xi}$ tend to zero as ξ tends to infinity, it follows from (20), (34), (35) and (43) that

$$(u_1, v_1, w_1) = \omega^{-1}(kp_1, -\phi_y p_{1\xi}, -\phi_z p_{1\xi}) \ . \tag{47}$$

Finally, we shall obtain a solution of η_2 in terms of p_1 and A_2. From (26), (37), (38) and (43), η_2 satisfies

$$\eta_{2yy} - (1/T)\eta_2 = -\ \nu_1 p_{1\xi\xi} + (1/T)\phi(z = d) p_{1\xi\xi} - (1/T)A_2 \ , \tag{48}$$

$$\eta_2 = 0 \quad \text{at} \quad z = d, \quad y = -b_1, b_2 \ . \tag{49}$$

We express η_2 as

$$\eta_2 = \nu_2(x,y)p_{1\xi\xi} + \nu_1 A_2 \tag{50}$$

where ν_2 satisfies

$$\nu_{2yy} - (1/T)\nu_2 = -\nu_1 + (1/T)\phi(z = d) , \tag{51}$$

$$\nu_2 = 0 \quad \text{at} \quad y = -b_1, b_2 . \tag{52}$$

Now we are in a position to derive the equation of the K-dV type. From (32) and (33), it follows that

$$v_{2y} + w_{2z} = -(k/\omega)(kp_{2\xi} + ku_1u_{1\xi} + u_{1t} + p_{1x}) - u_{1x} , \tag{53}$$

We integrate (53) over a cross section D, make use of the divergence theorem, (20), (28), (36), (39), (43), (47) and (50), and obtain

$$\begin{aligned} &-(k/\omega)p_1 \int_\Gamma h_x(h_y^2 + h_z^2)^{-1/2}ds + \int_{-b_1}^{b_2} [-\omega(\nu_2 p_{1\xi\xi\xi} + \nu_1 A_{2\xi}) \\ &+ k^2\omega^{-1}p_1\nu_1 p_{1\xi} + \omega^{-1}\phi_{zz}(z = d)\nu_1 p_1 p_{1\xi} + \nu_1 p_{1t}]dy \\ &= -\iint_D (k/\omega)[k(-\phi p_{1\xi\xi\xi} + A_{2\xi}) + k^3\omega^{-2}p_1p_{1\xi} \\ &+ (k/\omega)p_{1t} + p_{1x}]dydz - \iint_D (kp_1/\omega)_x dydz \end{aligned}$$

By rearranging the terms, the above equation reduces to

$$m_0 p_{1t} + m_1 p_{1x} + m_2 p_1 + m_3 p_1 p_{1\xi} + m_4 p_{1\xi\xi\xi} = 0 , \tag{54}$$

which is our main result, where

$$\begin{aligned} &m_0 = 2a(G)^{-2}, \quad m_1 = 2a(G)^{-1} , \\ &m_2 = -(G)^{-1} \int_\Gamma h_x(h_y^2 + h_z^2)^{-1/2}ds + a[(G)^{-1}]_x , \\ &m_3 = 3ka(G)^{-3} + \omega^{-1}\int_{-b_1}^{b_2} \phi_y(z = d)\nu_{1y}dy , \\ &m_4 = -\omega\int_{-b_1}^{b_2} \nu_2 dy - kG^{-1}\iint_D \phi dA , \end{aligned} \tag{55}$$

$\int_\Gamma h_x(h_y^2 + h_z^2)^{-1/2}ds$ is the line integral over the boundary of $h = 0$ in each cross section (Figure 1), and (h_x, h_y, h_z) is in the outward normal direction. Along a ray we let $\sigma = t$ and

$$d/d\sigma = \partial/\partial t + (dx/dt)\ \partial/\partial t = \partial/\partial t + G(x)\ \partial/\partial t \ .$$

In terms of σ, (54) becomes

$$m_0 P_{1\sigma} + m_2 P_1 + m_3 P_1 P_{1\xi} + m_4 P_{1\xi\xi\xi} = 0 \ . \tag{56}$$

We may also use x as a variable along a ray, then (56) assumes the form

$$m_1 P_{1x} + m_2 P_1 + m_3 P_1 P_{1\xi} + m_4 P_{1\xi\xi\xi} = 0 \ . \tag{57}$$

§4. A RECTANGULAR CHANNEL WITH VARIABLE WIDTH

We consider a symmetric rectangular channel with constant depth and variable width. Each cross section D of the channel is bounded by $z = 0$, $z = 1$, $y = \pm\ell(x)$, where $\ell(x)$ is a smooth function satisfying $0 < \ell_1 \leq \ell(x) \leq \ell_2$, and ℓ_1, ℓ_2 are two constants. The equations for ϕ, (44) to (46), now become

$$\nabla^2\phi = k^2 \qquad \text{in } D ,$$

$$\phi_z = \omega^2\nu_1 \qquad \text{at } z = 1, \ -\ell < y < \ell ,$$

$$\phi_y = 0 \qquad \text{at } y = -\ell, \ell, \ 0 < z < 1 ,$$

$$\phi_z = 0 \qquad \text{at } z = 0, \ -\ell < y < \ell ,$$

Let $\phi = \varphi + k^2z^2/2$. Then φ satisfies

$$\nabla^2\varphi = 0, \qquad \text{in } D ,$$

$$\varphi_z = -k^2 + \omega^2\nu_1, \qquad \text{at } z = 1, \ -\ell < y < \ell ,$$

$$\varphi_y = 0, \qquad \text{at } y = -\ell, \ell, \ 0 < z < 1 ,$$

$$\varphi_z = 0, \qquad \text{at } z = 0, \ -\ell < y < \ell .$$

which may be solved by separation of variables, where by (27)

$$\nu_1 = 1 - \cosh\mu y/\cosh\mu\ell , \tag{58}$$

and k, ω are assumed to be given by (31).

By symmetry, we may consider the problem in $0 \leq y \leq \ell$ and prescribe the condition $\varphi_y = 0$ at $y = 0$, $0 < z < 1$. It is easily found that

$$\phi = k^2z^2/2 + \sum_{n=0}^{\infty} A_n \cosh n\pi z/\ell \cos n\pi y/\ell , \tag{59}$$

$$A_n = (-1)^{n+1} 2\mu\omega^2 \tanh\mu\ell/[n\pi(\mu^2 + (n\pi/\ell)^2)\sinh n\pi/\ell], \quad n \geq 1 ,$$

where A_0 is arbitrary and we choose $A_0 = -k^2/2 + T$. Next we proceed to find the solution for ν_2. From (51), (52), (58) and (59), we have

$$\nu_{2yy} - \mu^2\nu_2 = \cosh\mu y/\cosh\mu\ell + \mu^2 \sum_{n=1}^{\infty} A_n \cosh n\pi/\ell \cos n\pi y/\ell, \quad -\ell < y < \ell ,$$

$$\nu_2 = 0 \quad \text{at} \quad y = -\ell, \ell .$$

By symmetry again, we may solve ν_2 for the half interval $0 < y < \ell$ and prescribe $\nu_{2y} = 0$ at $y = 0$. We find that

$$\nu_2 = -\ell \tanh \mu\ell \cosh \mu y/[2\mu \cosh \mu\ell] + y \sinh \mu y/[2\mu \cosh \mu\ell]$$
$$+ \sum_{n=1}^{\infty} B_n((-1)^n \cosh \mu y/\cosh \mu\ell - \cos n\pi y/\ell) . \tag{60}$$

where

$$B_n = \mu^2 A_n \cosh n\pi/\ell \; / \; [\mu^2 + (n\pi/\ell)^2] .$$

The expression for ϕ and ν_2 given by (59) and (60) enable us to evaluate the coefficients in (54) for a channel of constant depth but variable width. To be definite, we choose the positive sign for G. It is obtained from (55) that

$$m_0 = 4\ell[1 - (1/\mu\ell)\tanh \mu\ell] ,$$

$$m_1 = 2\ell[1 - (1/\mu\ell)\tanh \mu\ell]^{1/2} ,$$

$$m_2 = (1/2)d\, m_1/dx ,$$

$$m_3 = 6k\ell[1 - (1/\mu\ell)\tanh \mu\ell]^{3/2}$$
$$+ \sum_{n=1}^{\infty} [4\omega n\pi/(\mu^3\ell^2)][1 + (n\pi/\ell\mu)^2]^{-2}\tanh^2\mu\ell \tanh n\pi/\ell ,$$

$$m_4 = -(2\ell^2k^3/3)[1 - (1/\mu\ell)\tanh \mu\ell]^{1/2}$$
$$+ (\omega\ell/\mu^2)(\tanh^2\mu\ell - 1) + (\omega/\mu^3)\operatorname{tank} \mu\ell$$
$$+ \sum_{n=1}^{\infty} (4\omega^3\mu^2/n\pi)[\mu^2 + (n\pi/\ell)^2]^{-2}\tanh \mu\ell$$

Here we keep k and ω in m_3 and m_4 since other choices of S are also possible. We also remark in passing that all termwise differentiations and integrations needed in the derivation of m_0 to m_4 can be justified without difficulty, and m_4 is independent of the arbitrary coefficient A_0 in ϕ because of (28).

§5. DISCUSSION

We observe from the Hamilton-Jacobi equation (28) that the surface tension of the liquid has the apparent effect of reducing the width and increasing the mean depth of the channel. Therefore, the group velocity of a capillary wave is always greatr than that of a surface wave without surface tension, and is an increasing function of the nondimensional surface tension coefficient T. One solution for (28) is given in (30). On the other hand, we may prescribe $S = x$ at $t = 0$. Then S is determined implicitly by

$$\int_S^x [G(\zeta)]^{-1} d\zeta = t.$$

In this case,

$$\omega = G(S), \quad k = G(S)/G(x) .$$

The disadvantage of determining S implicitly may be compensated by the fact that the initial condition for (54) at $t = 0$ can be directly expressed in terms of $\xi = \beta^{-1}S$. Certainly one can also solve (28) by prescribing data on any simple smooth curve $G : t_0 = t_0(s), \quad x_0 = x_0(s)$, in the t,x-plane where we may identify S with s. Then S satisfies

$$\int_{x_0(S)}^x [G(\eta)]^{-1} d\eta = t - t_0(S) .$$

It is shown that the rays do not intersert each other if G has no characteristic direction.

We may derive the so-called Green's law (Lamb, 1932) for the change of wave amplitude as follows. If we linearized the governing equations, and applied the ray method (Keller (1958), we would obtain from (56)

$$m_1 p_{1x} + m_2 p_1 = 0$$

along a ray. Therefore,

$$p_1 = (p_1)_0 \exp[- \int_{x_0}^x (m_2/m_1) d\zeta] ,$$

where $(p_1)_0$ is the initial value of p_1, and by (26)

$$\eta_1 = \nu_1(x,y)(p_1)_0 \exp[-\int_{x_0}^{x} (m_2/m_1)d]] \quad . \tag{60}$$

We note that p_1, η_1 should be interpreted respectively as the wave amplitude of the first order pressure and surface elevation. The exponential in (60) will be called the amplification factor. As seen from (55), m_2 and m_1 do not depend upon the solution ϕ of the Neumann problem posed by (44) to (46), and can always be determined for a given channel. For a rectangular channel with variable width $b(x)$ and depth $d(x)$, it is found that

$$\eta_1 = \nu_1(x,y)(p_1)_0[b(1 - (1/\mu b)\tanh \mu b/2)]^{-1/2} d^{-1/4} \quad .$$

We see that the amplification factor is always greater for a surface wave with surface tension.

If the channel has a uniform cross section, then $h_x = (G^{-1})x = 0$ and from (55) $m_2 = 0$, and all other coefficients are constant. Furthermore, the rays are straight lines and given by $G^{-1}x - t = $ constant. We obtain from (50) that

$$m_0 p_{1\sigma} + m_3 p_1 p_{1\xi} + m_4 p_{1\xi\xi\xi} = 0 \ ,$$

from which expressions for progressive waves of permanent type can be found. Finally if T tends to zero, that is, μ tends to infinity, (55) reduce to those obtained in Zhong and Shen (1982) for waves without surface tension.

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An equation of the K-dV type is derived for the study of nonlinear surface waves under gravity with surface tension and edge constraints in a channel of variable cross section. The approach used here is based upon a nonlinear ray method. An example of a symmetric rectangular channel with variable width is given.

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